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Block-transitive, point-imprimitive designs with $\lambda = 1$

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Abstract

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Let $\mathscr{D}$ be a 2-$(v, k, 1)$ design with a group G of automorphisms which is transitive on the blocks of $\mathscr{D}$ and transitive but imprimitive on the points of $\mathscr{D}$. Delandtsheer and Doyen (1989) proved that v is bounded above by $(k-2)^2(k+1)^2/4$. Carrying on from the work of Cameron and Praeger (1989), we show that if v is equal to this upper bound then $v = 729$ and $k = 8$. Further work of Nickel et al. (1992) has shown that, up to an isomorphism, there are 467 block-transitive, point-imprimitive 2-$(729, 8, 1)$ designs.

1. Introduction

A 2-(v, k, λ) *design* $\mathscr{D}$ is a set Ω of v *points* and a set $\mathscr{B}$ of k-element subsets of Ω, called *blocks*, such that each pair of points is contained in exactly λ blocks. Such a design is called *block-transitive* if some group G of automorphisms of $\mathscr{D}$ is transitive on the block set $\mathscr{B}$. In this case G is transitive on the point set Ω by Block's lemma (see [6, 2.3.1]). It may happen that G preserves a nontrivial equivalence relation ρ on Ω: we say that G *preserves* ρ if two points α and β are ρ-related if and only if, for all $g \in G$, α^g and β^g are ρ-related. Also ρ is nontrivial if the ρ-classes $\rho(\alpha) := \{\beta \mid \alpha \rho \beta\}$ have size $c = |\rho(\alpha)|$ satisfying $1 < c < v$. If G does preserve such a nontrivial equivalence relation then G is said to act *point-imprimitively* on $\mathscr{D}$. The starting point for this paper is the recent result of Delandtsheer and Doyen [5] (see also [3, Theorem 5.1]), which shows that the number of points of a block-transitive, point-imprimitive 2-(v, k, λ) design $\mathscr{D}$ is bounded above by a function of the block size, namely,

$$v \leq (\tbinom{k}{2} - 1)^2.$$

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More precisely, if G is a group of automorphisms of $\mathscr{D}$ acting block-transitively and preserving a nontrivial equivalence relation ρ on points with d classes of size $c=v/d$, then

$$c=\frac{\binom{k}{2}-x}{y} \quad \text{and} \quad d=\frac{\binom{k}{2}-y}{x},$$

where x and y are positive integers. Here x is the number of unordered pairs of points in a given block of $\mathscr{D}$ which are ρ-related: since G is transitive on blocks, this number x is independent of the block. Note that the upper bound on v is independent of the number λ of blocks containing a given pair of points. The case where $v=(\binom{k}{2}-1)^2$ was examined by Cameron and Praeger [3] and a classification of all such designs was obtained provided that $k\neq 3,4,5$ or 8. In all of the examples found, the parameter λ was very large.

On the other hand, there are block-transitive, point-imprimitive designs with $\lambda=1$. For example, there are several examples with 91 points discussed in [4]. In this paper we concentrate on block-transitive, point-imprimitive 2-$(v,k,1)$ designs such that each block contains exactly one pair of points in the same equivalence class, that is, the parameter x above equals 1. One family of examples of such designs are the Desarguesian projective planes of order q, where $q\equiv 1 \pmod 3$.

Example. Let $\mathscr{D}$ be the design of points and lines of the projective plane $\mathrm{PG}(3,q)$, where $q\equiv 1 \pmod 3$. Then $\mathscr{D}$ is a 2-$(q^2+q+1, q+1, 1)$ design and admits a cyclic subgroup $G=\langle g\rangle$ of order q^2+q+1 of automorphisms which acts regularly on points and on lines (blocks). Thus, the points of $\mathscr{D}$ may be labelled by the elements of G in such a way that G acts by right multiplication. Since $q\equiv 1 \pmod 3$, G has a subgroup $K=\langle h\rangle$, where $h=g^{(q^2+q+1)/3}$, of order 3 and G preserves the equivalence relation ρ on points defined by: points a,b are ρ-related if and only if $aK=bK$. The ρ-classes are the cosets of K in G. Thus, G is transitive on blocks and imprimitive on the points of $\mathscr{D}$. Further, the orbit of G on unordered pairs of points containing a given pair, say $\{1,\alpha\}$, has order q^2+q+1 and consists of all the unordered pairs of ρ-related points. Since G is regular on the lines of $\mathscr{D}$, it follows that each line of $\mathscr{D}$ contains exactly one pair of ρ-related points. (The largest subgroup of $\mathrm{Aut}\,\mathscr{D}=\mathrm{P\Gamma L}(3,q)$ which preserves ρ is the normalizer in $\mathrm{Aut}\,\mathscr{D}$ of $\langle g\rangle$.)

Now let $\mathscr{D}$ be a nontrivial 2-$(v,k,1)$ design which admits a group G of automorphisms acting transitively on blocks and preserving a nontrivial equivalence relation ρ on points such that each block contains a unique ρ-related pair of points. Let K be the subgroup of G which fixes each ρ-class setwise. It is certainly possible for K to be nontrivial, as was seen in the family of examples above, where a cyclic group G of order v was given with the subgroup $K\simeq Z_3$ acting semiregularly on ρ-classes. Moreover, for the design $\mathrm{PG}(2,4)$, the subgroup of the full automorphism group $\mathrm{P\Gamma L}(3,4)$ which preserves ρ is isomorphic to $F_{21}\times S_3$, where F_{21} is the Frobenius group of order 21, and the subgroup of this which fixes every ρ-class setwise is S_3. It turns out that this is the only example for which the subgroup K can have even order.

Theorem 1.1. *Let $\mathscr{D}$ be a 2-$(v, k, 1)$ design with a group G of automorphisms which is transitive on the blocks of $\mathscr{D}$. Suppose that G preserves a nontrivial equivalence relation ρ on points with d classes of size c such that each block of $\mathscr{D}$ contains exactly one pair of ρ-related points. Let K be the subgroup of G fixing each ρ-class setwise. Then one of the following holds:*

(i) *The group K is trivial.*

(ii) *The group K is elementary abelian of odd order c, and is transitive on each ρ-class. Also d is odd. Moreover, if $c = 3$ then $\mathscr{D}$ is the design of points and lines of a projective plane.*

(iii) *$\mathscr{D}$ is* PG(2, 4), *$K \simeq S_3$ and G is $H \times K$, where H is Z_7 or a Frobenius group F_{21} of order* 21.

This result, together with [3, Corollary 5.4], allows us to classify the block-transitive, point-imprimitive 2-$(v, k, 1)$ designs attaining the Delandtsheer–Doyen bound. In this paper we prove the following theorem.

Theorem 1.2. *Let $\mathscr{D}$ be a 2-$(v, k, 1)$ design with a group G of automorphisms which acts block-transitively and point-imprimitively on $\mathscr{D}$, and for which $v = (\binom{k}{2} - 1)^2$. Then $\mathscr{D}$ is a* 2-(729, 8, 1) *design. Moreover, $G = NH$, where H is either cyclic of order* 13 *or the nonabelian group of order* 39, *and N satisfies one of the following:*

(a) $N = Z_3^6$,

(b) $N = Z_9^3$, *or*

(c) *N is the relatively free,* 3*-generator, exponent* 3, *nilpotency class* 2 *group (of order* 729).

The subgroup N of G acts regularly on the point set Ω; so, Ω can be identified with N in such a way that N acts by right multiplication. Further, if α is the point identified with the identity element of N, then the subgroup H of G can be chosen to be G_α (by replacing H by a conjugate if necessary). Moreover, the group $N \cdot Z_{13}$ acts transitively on the blocks of $\mathscr{D}$; so, for the purposes of investigating or classifying these designs, we may assume that $|H| = 13$. Then G has 28 orbits on unordered pairs of points, each of the same length $3^6 \cdot 13$. It follows from [3, Proposition 1.3] that, for an 8-element subset B of $\Omega = N$, the set $\{B^g \mid g \in G\}$ of images of B under G is the block set of a block-transitive, point-imprimitive 2-(729, 8, 1) design if and only if B contains an unordered pair of points from each of the G-orbits on unordered pairs. The search for 8-element subsets of N with this property was carried out for each of the three kinds of groups N and 467 pairwise nonisomorphic designs were found. This work was done by the authors together with Nickel and Niemeyer and is described in [10].

In Section 2 some general technical lemmas are proved about block-transitive, point-imprimitive designs. These are used in Section 3 to prove Theorem 1.1. Then, in Section 4, Theorem 1.2 is proved.

2. Block-transitive, point-imprimitive designs

Let $\mathscr{D}$ be a 2-(v,k,λ) design with a group G of automorphisms which is transitive on blocks and transitive but imprimitive on the point set Ω of $\mathscr{D}$. Suppose that G preserves a nontrivial equivalence relation ρ on Ω with d classes of size c, where $v=cd$, $c>1$, $d>1$. Then, by [5] (or see [3]),

$$c=(l-x)/y, \qquad d=(l-y)/x,$$

where $l=k(k-1)/2$, x and y are positive integers and x is the number of unordered pairs of ρ-related points in a block of $\mathscr{D}$. Let R be the set of ρ-classes. First we examine the action of G on R.

Lemma 2.1. *Let $\mathscr{D}$ and G be as above. Let Δ be an orbit of G in $R\times R$, $\Delta\neq\{(r,r)\mid r\in R\}$ and let $S(\Delta)=\{\{\alpha,\beta\}\mid\alpha\in r_1,\beta\in r_2\text{ for some }(r_1,r_2)\in\Delta\}$. Then the cardinality u of the set $\Delta(r):=\{r'\mid(r,r')\in\Delta\}$ is independent of $r\in R$, and each block of $\mathscr{D}$ contains*

$$\frac{2cux}{\delta(c-1)}$$

elements of $S(\Delta)$, where δ is 2 if Δ is self-paired and 1 if Δ is not self-paired. In particular, $2ux$ is divisible by $c-1$.

Recall that, for a transitive group G on a set R, an orbit Δ of G in $R\times R$ has a corresponding paired orbit $\Delta^*=\{(r_2,r_1)\mid(r_1,r_2)\in\Delta\}$ and Δ is said to be self-paired if $\Delta=\Delta^*$. In the case where $c=d=l-1$, we conclude that G acts *2-homogeneously* on R, that is, G is transitive on unordered pairs of R.

Corollary 2.2. *If $\mathscr{D}$ and G are as above and if $c=d=l-1$, then G is 2-homogeneous on R.*

Proof of Lemma 2.1. We have $|S(\Delta)|=c^2du/\delta$. Let N be the number of ordered pairs $(\{\alpha,\beta\},B)$, with B a block of $\mathscr{D}$ containing $\{\alpha,\beta\}$ and $\{\alpha,\beta\}\in S(\Delta)$. Then $N=|S(\Delta)|\lambda=c^2du\lambda/\delta$. Now, as G fixes $S(\Delta)$ setwise and G is transitive on blocks of $\mathscr{D}$, each block of $\mathscr{D}$ contains exactly N/b pairs from $S(\Delta)$, where b is the number of blocks. The rest of the proof is arithmetic: $b=v(v-1)\lambda/k(k-1)=cd(cd-1)\lambda/2l$. We have $cd-1=((l-x)(l-y)/xy)-1=l(l-x-y)/xy=l(c-1)/x$; so $b=cd(c-1)\lambda/2x$. Thus, $N/b=2cux/\delta(c-1)$. □

Proof of Corollary 2.2. If $c=d=l-1$ then $x=y=1$ and, by Lemma 2.1, $2uc/\delta(c-1)$ is an integer, for each nondiagonal orbit Δ of G in $R\times R$. In particular, $c-1$ divides $2u$. If $u=c-1$ then G is 2-transitive and, hence, 2-homogeneous on R. So, assume that $u<c-1$. Then $u=(c-1)/2$, so c is odd and, hence, $\delta=1$. It follows that the $c(c-1)/2$ ordered pairs in Δ correspond to $c(c-1)/2$ distinct unordered pairs of ρ-classes; hence, G is transitive on the unordered pairs from R, that is, G is 2-homogeneous on R. □

Now we shall restrict the discussion to 2-$(v,k,1)$ designs where $x=1$.

Lemma 2.3. *Let $\mathscr{D}$ be a 2-$(v,k,1)$ design, let $\mathscr{D}$ and G be as above and let $x=1$. Then the following hold:*

(a) *The group induced on a ρ-class is 2-homogeneous.*

(b) *If B is a block of $\mathscr{D}$ and (α,β) is the unique pair of ρ-related points in B, then $G_{\{\alpha,\beta\}}=G_B<G_{\rho(\alpha)}$, where $\rho(\alpha)$ is the ρ-class containing α.*

(c) *The number of blocks of $\mathscr{D}$ is $b=cd(c-1)/2$.*

Proof. Let α,β be a ρ-related pair of points of Ω and let B be the unique block containing α and β. By the uniqueness of B, $G_{\{\alpha,\beta\}}\subseteq G_B$, and, since B contains a unique pair of ρ-related points, it follows that $G_B=G_{\{\alpha,\beta\}}$. Consequently, $G_B\subseteq G_{\rho(\alpha)}$.

Let $\{\alpha,\beta\}$ and $\{\alpha',\beta'\}$ be two unordered pairs from $\rho(\alpha)$ and let B,B' be the (unique) blocks of $\mathscr{D}$ containing $\{\alpha,\beta\}$ and $\{\alpha',\beta'\}$, respectively. Let $g\in G$ map B to B'. Then, as $\{\alpha,\beta\},\{\alpha',\beta'\}$ is the unique ρ-related pair of points of B,B', respectively, $\{\alpha,\beta\}^g=\{\alpha',\beta'\}$. Moreover, $\rho(\alpha)^g\cap\rho(\alpha)$ contains $\{\alpha,\beta\}^g=\{\alpha',\beta'\}$ and, hence $\rho(\alpha)^g=\rho(\alpha)$, that is, $g\in G_{\rho(\alpha)}$. It follows that $G_{\rho(\alpha)}$ is 2-homogeneous on $\rho(\alpha)$.

It was shown in the proof of Lemma 2.1 that the number b of blocks of $\mathscr{D}$ is $b=cd(c-1)\lambda/2x=cd(c-1)/2$. □

From these results it is clear that we shall need to consider finite 2-homogeneous permutation groups. It emerged in the proof of Corollary 2.2 that a 2-homogeneous group G is either 2-transitive or has two nontrivial orbits Δ and Δ^* on ordered pairs which are paired with each other. In the latter case (see [8, XII 6.5]), G has odd order, its degree is a prime power $q\equiv 3 \pmod 4$, and G is isomorphic to a subgroup of $\Gamma L(1,q)$. Note that this result uses the Feit–Thompson theorem [7], that is, the theorem that all groups of odd order are soluble. The following lemma establishes some properties of 2-homogeneous groups we shall need. Recall that, for a finite transitive permutation group G on a set Ω, each orbit of G_α, where $\alpha\in\Omega$, is of the form $\Delta(\alpha)=\{\beta\mid(\alpha,\beta)\in\Delta\}$, for some G-orbit Δ in $\Omega\times\Omega$. Moreover, for paired orbits Δ and Δ^* in $\Omega\times\Omega$, the corresponding G_α-orbits $\Delta(\alpha)$ and $\Delta^*(\alpha)$ are also said to be paired and their lengths are equal: $|\Delta(\alpha)|=|\Delta|/|\Omega|=|\Delta^*|/|\Omega|=|\Delta^*(\alpha)|$.

Lemma 2.4. *Let G be a 2-homogeneous permutation group on a set Ω of c points and let K be a nontrivial normal subgroup of G. Then the following hold:*

(a) *K is transitive on Ω.*

(b) *For $\alpha\in\Omega$, all K_α-orbits in $\Omega-\{\alpha\}$ have the same length, say m.*

(c) *All K-orbits on unordered pairs of points from Ω have the same length, say n. If $|K|$ is odd then c is a prime power, $n=cm$, and, for distinct points α and β of Ω, $K_{\{\alpha,\beta\}}=K_{\alpha,\beta}$. If $|K|$ is even then $n=cm/2$ (and all K-orbits in $\Omega\times\Omega$ are self-paired).*

Proof. It is straightforward to show that a 2-homogeneous group G is primitive. Hence, a nontrivial normal subgroup K of G is transitive on Ω. Let $\alpha\in\Omega$. Then K_α is a normal subgroup of G_α. If G_α is transitive on $\Omega-\{\alpha\}$ then G_α permutes the K_α-orbits in $\Omega-\{\alpha\}$ transitively and, so, all K_α-orbits in $\Omega-\{\alpha\}$ have the same length m say. If this is not the case then G has two nontrivial orbits, Δ and Δ^*, on ordered pairs which are paired with each other. Thus, G_α has two orbits, $\Delta(\alpha)=\{\beta\,|\,(\alpha,\beta)\in\Delta\}$ and $\Delta^*(\alpha)=\{\gamma\,|\,(\alpha,\gamma)\in\Delta^*\}=\{\gamma\,|\,(\gamma,\alpha)\in\Delta\}$ in $\Omega-(\alpha)$. Now G_α permutes transitively the K_α-orbits in $\Delta(\alpha)$ and, so all of these K_α-orbits have the same length, say m, and, similarly, all the K_α-orbits in $\Delta^*(\alpha)$ have the same length, say m^*. Let $\Sigma(\alpha)$ be a K_α-orbit in $\Delta(\alpha)$ and let $\gamma\in\Sigma(\alpha)\subseteq\Delta(\alpha)$. Then $(\alpha,\gamma)\in\Delta$; so $\Sigma=(\alpha,\gamma)^K\subseteq\Delta$, whence $\Sigma^*\subseteq\Delta^*$ and, so $\Sigma^*(\alpha)\subseteq\Delta^*(\alpha)$. It follows that $m=|\Sigma(\alpha)|=|\Sigma^*(\alpha)|=m^*$.

Since G is transitive on unordered pairs, G permutes transitively the set of K-orbits on unordered pairs and, so these all have the same length, say n. If $|K|$ is odd then, by [13, 16.5], K has no nontrivial self-paired orbits in $\Omega\times\Omega$ and, so, if α and β are distinct points of Ω then the orbit $(\alpha,\beta)^K$ of K in $\Omega\times\Omega$ has the same length as the K-orbit on unordered pairs containing $\{\alpha,\beta\}$; so $K_{\{\alpha,\beta\}}=K_{\alpha,\beta}$. Thus, $n=|(\alpha,\beta)^K|=cm$. Moreover, if $|K|$ is odd then K is soluble [7] and, hence, a minimal normal subgroup of G contained in K is elementary abelian and regular on Ω; so c is a prime power. Suppose now that $|K|$ is even so that, by [14, 16.5], K has a self-paired orbit, say Σ, in $\Omega\times\Omega$. Then the K-orbit on unordered pairs consisting of those pairs $\{\alpha,\beta\}$ for which $(\alpha,\beta)\in\Sigma$ has length $|\Sigma|/2$, since each unordered pair $\{\alpha,\beta\}$ corresponds to the two ordered pairs (α,β) and (β,α) of Σ. Thus, in this case $n=cm/2$. It follows that all K-orbits in $\Omega\times\Omega$ are self-paired.

3. Block-transitive, point-imprimitive 2-$(v,k,1)$ designs with $x=1$

In this section we prove Theorem 1.1. The main part of the proof is contained in Proposition 3.1; then the cases $c=3$ and $c=5$ are considered separately.

Proposition 3.1. *Let $\mathscr{D}$ be a 2-$(v,k,1)$ design with a group G of automorphisms acting transitively on blocks. Suppose that G preserves a nontrivial equivalence relation ρ on points with d classes of size c such that any block of $\mathscr{D}$ contains a unique pair of ρ-related points. Suppose further that the subgroup K of G fixing each ρ-class setwise is nontrivial. Then $v=cd$ is odd and either*

(a) *K is elementary abelian of order c and is transitive on each ρ-class, or*

(b) *the 4-tuple (c,d,k,K) is $(3,7,5,S_3)$ or $(5,5,4,D_{10})$.*

Proof. Let Ω denote the point set of $\mathscr{D}$ and let $\alpha\in\Omega$. By Lemmas 2.3 and 2.4, $G_{\rho(\alpha)}$ is 2-homogeneous on $\rho(\alpha)$ and, as K is normal in $G_{\rho(\alpha)}$, K is transitive on $\rho(\alpha)$ and all K_α-orbits in $\rho(\alpha)-\{\alpha\}$ have the same length, say m. Thus, m divides $c-1$. Also, since G is transitive on blocks and K is normal in G, all K-orbits on blocks have the same length, say n. Then n divides $b=dc(c-1)/2$. Now let $\beta\in\rho(\alpha)-\{\alpha\}$ and let $B(\alpha,\beta)$ be the unique block of $\mathscr{D}$ containing α and β. Then $n=|K:K_{B(\alpha,\beta)}|=|K:K_{\{\alpha,\beta\}}|$ is cm if $|K|$ is odd and $cm/2$ if $|K|$ is even, by Lemma 2.4 applied to $G_{\rho(\alpha)}^{\rho(\alpha)}$, noting that $|K|$ and $|K^{\rho(\alpha)}|$ have the same parity.

Now let r_1, r_2 be distinct ρ-classes, and consider the action of K on $r_1 \times r_2$. Let $(\gamma,\delta)\in r_1\times r_2$ and let B be the unique block of $\mathscr{D}$ containing γ and δ. Then $K_{\gamma\delta}\subseteq K_B$ and, so the length of the K-orbit containing (γ,δ) is divisible by $n=|K:K_B|$. Suppose first that $|K|$ is odd. Then each K-orbit in $r_1\times r_2$ has length divisible by $n=cm$ and, hence, $|r_1\times r_2|/c=c$ is divisible by m. Since m divides $c-1$, it follows that $m=1$. Thus, $K_{\{\alpha,\beta\}}=K_{\alpha\beta}=K_\alpha$ fixes $\rho(\alpha)$ pointwise. So, $K^{\rho(\alpha)}$ is a regular normal subgroup of $G^{\rho(\alpha)}_{\rho(\alpha)}$, and, hence, $K^{\rho(\alpha)}$ is elementary abelian of order c (see [1, 8, XII 6.5]). We claim that $K_\alpha=1$. Let $\gamma\in\Omega-\rho(\alpha)$ and let $B(\alpha,\gamma)$ be the unique block containing α and γ. Then $K_{B(\alpha,\gamma)}$ has index c in K and fixes $\rho(\alpha)\cap B(\alpha,\gamma)$ a set of size 1 or 2, setwise. We have just shown that K-orbits on points or pairs of points from $\rho(\alpha)$ all have length c, and, so, the stabilizer of $\rho(\alpha)\cap B(\alpha,\beta)$ is K_α and is equal to $K_{B(\alpha,\gamma)}$. Similarly, $K_{B(\alpha,\gamma)}=K_\gamma$, whence $K_\alpha=K_\gamma=1$. Finally, $d=1+(c-1)y$ and so d, and hence v, is odd if c is odd. Thus, (a) is true.

Assume now that $|K|$ is even. If k were equal to 3 then we would have $c=(l-1)/y=2/y=2$, $d=2$ and, hence, $b=2<v$, which is not possible. Thus, $k\geqslant 4$. Then in a block B of the design there are points μ, ν for which $B\cap\rho(\mu)=\{\mu\}$ and $B\cap\rho(\nu)=\{\nu\}$. Then K_B fixes μ and ν, while $K_{\mu\nu}$ fixes the unique block B containing μ and ν; hence, $K_B=K_{\mu\nu}$. Thus, $|K:K_{\mu\nu}|=n=cm/2$ and, so, $|K_\mu:K_{\mu\nu}|=m/2$, whence m is even. From the previous paragraph, every K-orbit in $r_1\times r_2$ has length divisible by $n=cm/2$, whence m divides $2c$. Since also m divides $c-1$, it follows that $m=2$ and c is odd. Also $d=1+(c-1)y$ is odd; so, v is odd. Now consider K_α. All of its orbits in $\rho(\alpha)-\{\alpha\}$ have length 2. Let r be a ρ-class distinct from $\rho(\alpha)$. Let $\gamma\in r$ and let B' be the block containing α and γ. If $B'\cap\rho(\alpha)=\{\alpha\}$ then $K_{B'}\subseteq K_\alpha$, and, as $|K:K_{B'}|=n=c=|K:K_\alpha|$, we have $K_{B'}=K_\alpha$. Then the K_α-orbit containing γ has size $|r\cap B'|=1$ or 2. On the other hand, if $B'\cap\rho(\alpha)=\{\alpha,\alpha'\}$, say, then $B'\cap r=\{\gamma\}$ and, so, $K_{B'}=K_\gamma=K_{\{\alpha,\alpha'\}}$; hence, $K_\gamma\cap K_\alpha$ has index 2 in K_γ. It follows that the K_α-orbit containing γ has length 2. Thus, all K_α-orbits in r have lengths 1 or 2, and, as c is odd, K_α fixes some point of r. Therefore, K acts similarly on all ρ-classes. We may assume that $K_\alpha=K_\gamma$, where $\gamma\in r$. Then K_α has one orbit, $\{\gamma\}$, of length 1 and $(c-1)/2$ orbits of length 2 in r. If B' is the block containing α and γ then B' intersects at least one of $\rho(\alpha)$ and r in exactly one point, and it follows that $K_{B'}=K_\alpha=K_\gamma$ and $|B'\cap\rho(\alpha)|=|B'\cap r|=1$. Moreover, since $\{\gamma\}$ is the only K_α-orbit in r of length 1, the only block containing α which meets both $\rho(\alpha)$ and r in one point is B'. The number of triples (B',α',γ'), where B' is a block of $\mathscr{D}$, $B'\cap\rho(\alpha')=\{\alpha'\}$, $B'\cap\rho(\gamma')=\{\gamma'\}$ and $\alpha'\neq\gamma'$, is $b(k-2)(k-3)$. On the other hand, a point α and a class $r\neq\rho(\alpha)$ uniquely determine a block B and a point $\gamma\in r$ for which (B,α,γ) is such a triple. Hence, the number of triples is $v(d-1)$. Thus, $(k-2)(k-3)cd(c-1)/2=b(k-2)(k-3)=v(d-1)=cd(d-1)=cd(c-1)y$; whence, $2y=(k-2)(k-3)$. Then, as $c=(l-1)/y$, y divides $l-1=(k-2)(k+1)/2$ and it follows that k is 4, 5 or 7. However, as c is odd, $k\neq 7$. Thus, (c,d,k) is $(3,7,5)$ or $(5,5,4)$, and K is S_3 or D_{10}, respectively. This completes the proof of Proposition 3.1. □

Our next step is to consider the case where $c=3$ in Proposition 3.1.

Lemma 3.2. *Let $\mathscr{D}$ be as in Proposition* 3.1 *and let $c=3$. Then $\mathscr{D}$ is the design of points and lines of a projective plane of order $k-1$. Moreover, if $(c,d,k,K)=(3,7,5,S_3)$ then $\mathscr{D}$ is* PG(2,4) *and G is $H\times K$, where H is Z_7 or F_{21}.*

Proof. If $c=3$ then $l=1+3y$ and $d=l-y=(1+2l)/3=(k^2-k+1)/3$; so, $v=k^2-k+1$, whence $\mathscr{D}$ is the design of points and lines of a projective plane of order $k-1$. Now let $(c,d,k,K)=(3,7,5,S_3)$; so, $\mathscr{D}$ is the (unique) projective plane PG$(2,4)$ of order 4; see [6, 3.2.15].

Since the orbits of $K\simeq S_3$ are the ρ-classes, it follows that K contains a subgroup L of order 3 acting fixed-point-freely on the points of PG$(2,4)$. Now G normalizes L, and the normalizer of L in PΓL$(3,4)$ is the normalizer of a Singer cycle in PΓL$(3,4)$, which is isomorphic to $F_{21}\times S_3$. Since G is transitive on the lines of PG$(2,4)$ and G contains $K\simeq S_3$, it follows that G is $H\times K$, where H is Z_7 or F_{21}. $\square$

Remark. Conversely, suppose that $\mathscr{D}$ is a projective plane and G is a group of automorphisms of $\mathscr{D}$ which acts transitively on the lines of $\mathscr{D}$ and has a normal subgroup N of order 3. Then $\mathscr{D}$ satisfies the hypotheses of Proposition 3.1, with the ρ-classes being the orbits of N, and with $c=3$. However, it is a longstanding conjecture (see [6]) that a projective plane with a group of automorphisms transitive on the lines is Desarguesian. In other words, the known examples are the examples given in Section 1.

Lemma 3.3. *There is no design satisfying the hypotheses of Proposition* 3.1 *with* $(c,d,k,K)=(5,5,4,D_{10})$.

Proof. Let C be the centralizer of K in G. Then $H=C\times K$ has index 1 or 2 in G since $D_{10}\simeq KC/C\leqslant G/C\leqslant \mathrm{Aut}\,K$ and $\mathrm{Aut}\,D_{10}$ has order 20. Moreover, $K^{\rho(\alpha)}\simeq D_{10}$ is a normal self-centralizing subgroup of $G_{\rho(\alpha)}^{\rho(\alpha)}$ and is not equal to $G_{\rho(\alpha)}^{\rho(\alpha)}$ since the latter group is 2-homogeneous. Hence, $|G{:}H|=2$ and $G_{\rho(\alpha)}^{\rho(\alpha)}$ is a Frobenius group F_{20}.

Let α and β be ρ-related and let B be the block of $\mathscr{D}$ containing them, so that $G_B=G_{\{\alpha,\beta\}}$. Since K has two orbits of size 5 on unordered pairs of points in $\rho(\alpha)$, it follows that $G_{\{\alpha,\beta\}}K$ has index 2 in $G_{\rho(\alpha)}$ and, hence, that, in the action on the set R of ρ-classes, $G_B^R=G_{\{\alpha,\beta\}}K/K$ has index 2 in $G_{\rho(\alpha)}^R$. By Corollary 2.2, G^R is 2-homogeneous of degree 5 and, as $G_{\rho(\alpha)}^R$ has a subgroup G_B^R of index 2 which has at least 3 orbits in R, it follows that $G^R=F_{20}$ and C is D_{10}. Also $G=(K\times C)\cdot 2=(D_{10}\times D_{10})\cdot 2$. The set Ω of points may be identified with $O_5(G)=Z_5\times Z_5$, so that $O_5(G)$ acts by right multiplication and the ρ-classes are $r_i=\{i\}\times\mathbb{Z}_5$ for $i\in\mathbb{Z}_5$. Since $G_{\rho(\alpha)}^{\rho(\alpha)}$ is 2-transitive, we may assume that $\alpha=(0,1)$, $\beta=(0,4)$, and then $G_{\{\alpha,\beta\}}=\langle s,t\rangle$, where $(i,j)s=(i,-j)$ and $(i,j)t=(-i,j)$ for $(i,j)\in\Omega$. Since $G_{\{\alpha,\beta\}}$ fixes $B-\{\alpha,\beta\}$ setwise, it follows that $B-\{\alpha,\beta\}$ is either $\{(1,0),(4,0)\}$ or $\{(2,0),(3,0)\}$. In the former case however, translating B by $(1,4)\in O_5(G)$ gives a block $\{(1,0),(1,3),(2,4),(0,4)\}$ of $\mathscr{D}$ which is different from B but contains $(1,0)$ and $(0,4)$. This contradicts the fact that $\lambda=1$. Similarly, in the latter case, translating B by $(3,4)\in O_5(G)$ gives a block $\{(3,0),(3,3),(0,4),(1,4)\}$ of $\mathscr{D}$ different from B and containing $(3,0)$ and $(0,4)$, which is again a contradiction. Thus, there are no examples in this case and the lemma is proved. $\square$

Theorem 1.1 now follows from Proposition 3.1, and Lemmas 3.2 and 3.3.

4. 2-$(v, k, 1)$ designs with $v = ((k/2) - 1)^2$

Let $\mathscr{D}$ be a 2-$(v, k, 1)$ design with $v=(l-1)^2$, where $l=k(k-1)/2$. Suppose that G is a group of automorphisms of $\mathscr{D}$ which acts transitively on blocks and imprimitively on points. Then (see [3]), G preserves an equivalence relation ρ on the set Ω of points with d classes of size c, where $c=d=l-1$ (so, $x=y=1$). By [3, Corollary 5.4], k is $3, 4, 5$ or 8. If $k=3$ then $v=4$ and $\mathscr{D}$ is the trivial design on 4 points. But this is a 2-$(4, 3, 2)$ design; hence, k is $4, 5$ or 8 and c is $5, 9$ or 27, respectively. Let K be the subgroup of G which fixes each ρ-class setwise. By Theorem 1.1, either K is trivial or K is elementary abelian of order c and is transitive on each ρ-class. By Corollary 2.2, G is 2-homogeneous on the set R of ρ-classes and, by Lemma 2.3, $G_{\rho(\alpha)}$ is 2-homogeneous on $\rho(\alpha)$. Now $|R|=|\rho(\alpha)|=c$ and we note (see the remarks following Lemma 2.3) that all 2-homogeneous groups of degree 5 or 9 are 2-transitive. We shall show that in all cases $K \neq 1$ and the groups G^R and $G_{\rho(\alpha)}^{\rho(\alpha)}$ have elementary abelian regular normal subgroups.

Lemma 4.1. *The group K is elementary abelian of order c, and G^R and $G_{\rho(\alpha)}^{\rho(\alpha)}$ both have regular normal subgroups.*

Proof. Suppose first that $K=1$. Then G is isomorphic to a 2-homogeneous and, hence, primitive subgroup of S_c and $|G|$ is divisible by $v=c^2$. Hence, $c \neq 5$. If $c=9$ then G contains a Sylow 3-subgroup of S_9; hence, G contains a 3-cycle, so G is A_9 or S_9. However, in this case $G_{\rho(\alpha)}$ is A_8 or S_8 and, so, $G_{\rho(\alpha)}$ cannot act transitively on $\rho(\alpha)$. Thus, $c \neq 9$. If $c=27$ then, as $G_{\rho(\alpha)}$ is transitive on $\rho(\alpha)$ of degree 27, it follows that G does not contain A_{27}. Then (see [2, 8, XII 6.5]), $G \leqslant \mathrm{AGL}(3,3)$ and, so, $G_{\rho(\alpha)} \leqslant \mathrm{GL}(3,3)$. Since $|G_{\rho(\alpha)}|$ is divisible by 27 and since G^R is 2-homogeneous, it follows (see [9]) that $G_{\rho(\alpha)}$ is $\mathrm{SL}(3,3)$ or $\mathrm{GL}(3,3)$. But neither of these groups can act 2-homogeneously on $\rho(\alpha)$ of degree 27. Thus, K is nontrivial; so, by Theorem 1.1, K is elementary abelian of order c. Hence, $G_{\rho(\alpha)}^{\rho(\alpha)}$ has a regular normal subgroup, namely, $K^{\rho(\alpha)}$.

Let α and β be distinct ρ-related points and let B be the block of $\mathscr{D}$ containing $\{\alpha, \beta\}$. Then by Lemma 2.3, $G_B = G_{\{\alpha,\beta\}} \subseteq G_{\rho(\alpha)}$.

Now $|G_{\rho(\alpha)}:G_B| = c(c-1)/2$ and, hence, $|G_{\rho(\alpha)}:G_B K| = (c-1)/2$. Further, $G_B K$ fixes setwise the set of $k-2$ equivalence classes which meet B in one point. The only 2-homogeneous group G of degree 5 such that $G_{\rho(\alpha)}$ has a subgroup of index 2 with a fixed set of size 2 is the Frobenius group F_{20}; so, if $c=5$ then $G^R = F_{20}$. Similarly, the only 2-homogeneous subgroups G of S_9 such that $G_{\rho(\alpha)}$ has a subgroup of index 4 with a fixed set of size 3 (see [13]) are subgroups of $\mathrm{AGL}(2,3)$.

Finally, the only 2-homogeneous subgroups G of S_{27} such that $G_{\rho(\alpha)}$ has a subgroup of index 13 with a fixed set of size 6 (see [2, 8, XII 6.5]) are subgroups of $\mathrm{AGL}(3,3)$. $\square$

Now we show that the cases $c=5$ and $c=9$ do not arise.

Lemma 4.2. *There are no examples when $c=5$.*

Proof. By Lemma 4.1, $K \simeq Z_5$ and $G/K \simeq F_{20}$; so, $O_5(G)$ has order 25 and is regular on points. Thus, we may identify the point set Ω with $O_5(G)$ in such a way that $O_5(G)$ acts by right multiplication. Now if α is identified with the identity element of $O_5(G)$ then $G=O_5(G)G_\alpha$ and G_α acts by conjugation on Ω. Moreover, $\rho(\alpha)$ is a subgroup of $O_5(G)$ of order 5 which is fixed by K. Hence, $\rho(\alpha)=K$ and the other ρ-classes are the cosets of K in $O_5(G)$.

Suppose first that $O_5(G)$ is elementary abelian. Then G_α fixes at least two of the six subgroups of $O_5(G)$ of index 5, namely, K and H, say, and H is normal in G. Then G preserves a second nontrivial equivalence relation τ on Ω the equivalence classes of which are the cosets of H in $O_5(G)$. By [5] (or see [3]), each block of $\mathscr{D}$ contains a unique pair of τ-related points. We may take $O_5(G)=\{(s,t)\mid s,t\in\mathbb{Z}_5\}$, $K=\langle(1,0)\rangle$, $H=\langle(0,1)\rangle$. Let B be the block of $\mathscr{D}$ containing $(1,0)$ and $(4,0)$. Then G_B is the subgroup of G_α of order 2; it must invert K and, as $G_B^R=G_BK/K$ also has order 2, it must invert $O_5(G)/K\simeq H$. Since G_B normalizes H, it follows that $G_B=\langle\sigma\rangle$, where $(s,t)\sigma=(-s,-t)$. So, the other two points of B are (s,t) and $(-s,-t)$ for some s,t, with $t\neq 0$. Since B must contain exactly one pair of τ-related points, that is, exactly one pair of points with the same first entries, it follows that $s=0$, and we may take t to be 1 or 2. If $t=1$ then translating B by $(1,4)\in O_5(G)$ yields another block $\{(2,4),(0,4),(1,0),(1,3)\}$ containing $(0,4)$ and $(1,0)$, while if $t=2$ then translating B by $(1,2)$ yields a second block $\{(2,2),(0,2),(1,4),(1,0)\}$ containing $(0,2)$ and $(1,0)$. Thus, no design with $\lambda=1$ arises in this case.

Thus, we may identify $O_5(G)$ with the additive group $\mathbb{Z}_{25}$ of integers modulo 25, and G_α with the group $\langle h\rangle$, where $j^h\equiv 7j \pmod{25}$ for $j\in\mathbb{Z}_{25}$. The ρ-classes are $r_i=\{j\mid j\equiv i \pmod 5\}$, where $i=0,1,2,3,4$, and $r_0=K$. Let B be the block of $\mathscr{D}$ containing 5 and 20. Then $G_B=\langle h^2\rangle$. (Note that $j^{h^2}\equiv -j \pmod{25}$ for all $j\in\mathbb{Z}_{25}$.) Hence, the other two points of B are s and $-s$ for some $1\leqslant s\leqslant 12$, with $s\neq 5,10$. Translating $B=\{5,20,s,-s\}$ by $5-s\in\mathbb{Z}_{25}$ gives a block $B'=\{10-s,-s,5,5-2s\}$ of $\mathscr{D}$ containing 5 and $-s$ and, hence, $B=B'$. This means, however, that $10-s$ is equal to 20 or s, that is, s is equal to 15 or 5, respectively, which is a contradiction. Thus, there are no examples with $c=5$. $\square$

Lemma 4.3. *There are no examples when $c=9$.*

Proof. By Lemma 4.1, K is elementary abelian of order 9 and $G^R=G/K\leqslant \mathrm{AGL}(2,3)$. Moreover, G^R is 2-homogeneous and, as $9\equiv 1 \pmod 4$, G^R is 2-transitive. Thus $G=O_3(G)G_\alpha$, where $O_3(G)$ is regular on the 81 points of $\mathscr{D}$, and $G_\alpha\leqslant \mathrm{GL}(2,3)$ is transitive on $R-\{\rho(\alpha)\}$. A Sylow 2-subgroup of G_α is transitive on $R-\{\rho(\alpha)\}$ and on $\rho(\alpha)-\{\alpha\}$, both of degree 8. Since a Sylow 2-subgroup of $\mathrm{GL}(2,3)$ has order 16, and since the only subgroups of $\mathrm{GL}(2,3)$ of order 8 which are transitive are cyclic or quaternion, it follows that G_α has a subgroup H isomorphic to Z_8 or Q_8 such that H is transitive on $R-\{\rho(\alpha)\}$ and $\rho(\alpha)-\{\alpha\}$. Let $\bar{G}=O_3(G)H$. Then the subgroup of

$\bar{G}$ fixing a block of $\mathscr{D}$ containing two points β, γ of $\rho(\alpha)$ is $\bar{G}_{\{\beta,\gamma\}}$, a subgroup of $\bar{G}_{\rho(\alpha)}$ of order 2 (since $\bar{G}_{\rho(\alpha)}$ is sharply 2-transitive on $\rho(\alpha)$). Hence, $\bar{G}$ is transitive on blocks and we may, therefore, assume that $H = G_2$ and $\bar{G} = G$.

As usual, we may identify the point set Ω with $O_3(G)$ acting by right-multiplication. Identifying α with the identity of $O_3(G)$, we have $G_\alpha = H$ acting by conjugation. Then $\rho(\alpha) = K$. Now $H \simeq Z_8$ or Q_8 contains a unique involution z which is the central involution of $\mathrm{GL}(2,3)$ and z inverts K and $O_3(G)/K$. Let β, γ be two points of $\rho(\alpha) = K$ which are interchanged by z and let B be the block of $\mathscr{D}$ containing β and γ. Then $G_B = \langle z \rangle$. Now G_B should fix setwise the three ρ-classes which meet B in one point. But this is not the case since z interchanges the 8 classes in $R - \{\rho(\alpha)\}$ in pairs. Thus, no such design exists. $\square$

Thus, we have $c = 27$, $k = 8$ and, by Lemma 4.1, K is elementary abelian of order 27 and $G^R = G/K \leqslant \mathrm{AGL}(3,3)$. Thus, $G = O_3(G)G_\alpha$, with $O_3(G)$ acting regularly on the 729 points, and $G_\alpha \leqslant \mathrm{GL}(3,3)$. Let H be a Sylow 13-subgroup of G_α. Then $|H| = 13$ (since G^R is 2-homogeneous) and $O_3(G)H$ is transitive on the $729 \cdot 13$ blocks of $\mathscr{D}$. As usual, we may identify the point set Ω with $O_3(G)$ so that $O_3(G)$ acts by right multiplication and, identifying α with the identity of $O_3(G)$, G_α acts by conjugation. Then $\rho(\alpha) = K$ and the ρ-classes are the cosets of K in $O_3(G)$. Further, we know that both K and $O_3(G)/K$ are elementary abelian of order 27. We shall show that there are, up to an isomorphism, exactly three possibilities for the group $O_3(G)$. First we investigate the action of G on R in more detail. Note that, since the centre of $O_3(G)$ intersects K nontrivially, and since G_α acts irreducibly on K, the subgroup K is contained in the centre of $O_3(G)$.

Lemma 4.4. *The action of G on R is 2-homogeneous but not 2-transitive.*

Proof. Suppose to the contrary that G^R is 2-transitive. Then the group $G_\alpha \simeq G^R_{\rho(\alpha)}$ either contains $\mathrm{SL}(3,3)$ or is contained in $\Gamma\mathrm{L}(1,27)$ (see [9, Appendix]). We may identify K with a 3-dimensional vector space V over GF(3) in such a way that α is the zero vector. Let w be a nonzero vector of V and let B be the block of $\mathscr{D}$ containing $\{w, -w\}$. Suppose first that $G_\alpha \geqslant \mathrm{SL}(3,3)$. Then $G_B = G_{\{w,-w\}}$ is a subgroup of $G_{\rho(\alpha)}$ since $\{w, -w\} \subseteq \rho(\alpha)$. Now $G_B^{\rho(\alpha)} \leqslant G_{\rho(\alpha)}^{\rho(\alpha)} \leqslant \mathrm{AGL}(3,3)$ and G_B fixes the affine line containing $\{w, -w\}$, namely, $\langle w \rangle = \{0, w, -w\}$. It follows that G_B fixes α (identified with the zero vector) and, so, G_B is the subgroup of G_α (which is $\mathrm{GL}(3,3)$ or $\mathrm{SL}(3,3)$) fixing $\langle w \rangle$ setwise. Thus, in its action on R, with R also identified with V, G_B^R is the stabilizer in G^R of the zero vector (which is identified with $\rho(\alpha)$) and a subspace of dimension 1 or 2. Thus, the G_B-orbits on R have lengths either $1, 2, 24$ or $1, 8, 18$, respectively. Thus, it is impossible for G_B to fix setwise the set of six ρ-classes which meet B in one point. Hence, $G_\alpha \leqslant \Gamma\mathrm{L}(1,27)$ and, as G_α is transitive on $R - \{\rho(\alpha)\}$, G_α contains the central involution z of $\mathrm{GL}(3,3)$. Now z inverts $O_3(G)/K$ and, as G_α is also transitive on $\rho(\alpha) - \{\alpha\}$, z also inverts K. It is easy to check (since K is the centre of $O_3(G)$) that an involution with this property must invert each element of $O_3(G)$. Let

$\bar{G}=O_3(G)\langle H, z\rangle$. Then $\bar{G}$ is transitive on blocks (since $O_3(G)H$ is), and $\bar{G}_B=\langle z\rangle$. If $u\in B-\{w,-w\}$ then the inverse of u, which is u^z, is also in $B=B^z$. Thus, we have two pairs $\{w,u\}$ and $\{-w,u^z\}=\{w,u\}^z$ from B which lie in the same $\bar{G}$-orbit on unordered pairs of points. However, by [3, Proposition 1.3], if Δ is an orbit of $\bar{G}$ on unordered pairs of points and q_Δ is the number of unordered pairs of points of B which lie in Δ, then $q_\Delta/|\Delta|$ should be independent of Δ. Now the Sylow 2-subgroup $\langle z\rangle$ of $\bar{G}$ has normalizer in $\bar{G}$ equal to $\langle H,z\rangle\simeq Z_{26}$ and, hence, $\bar{G}$ contains 3^6 distinct conjugates of z. Each conjugate of z fixes $(3^6-1)/2$ unordered pairs of points setwise (since z inverts $O_3(G)=\Omega$) and no unordered pair is fixed setwise by more than one involution (since the stabilizer in $\bar{G}$ of an unordered pair has index in $\bar{G}$ divisible by $3^6\cdot 13$). Hence, each of the $3^6(3^6-1)/2$ unordered pairs of points is fixed setwise by an involution of $\bar{G}$ and it follows that $\bar{G}$ has 28 orbits on unordered pairs of points, each of length $3^6\cdot 13$. Thus, B should contain exactly one unordered pair of points from each of these $\bar{G}$-orbits. So, the fact that B contains two pairs from the same $\bar{G}$-orbit on pairs is a contradiction. □

Thus, G^R is a 2-homogeneous group of odd order. So, G_α is either H or $H\cdot 3$ and $G_\alpha\simeq G^R_{\rho(\alpha)}\leqslant \Gamma\mathrm{L}(1,27)$.

Lemma 4.5. *The group G is $O_3(G)G_\alpha$, where $G_\alpha\simeq Z_{13}$ or $G_\alpha\simeq Z_{13}\cdot Z_3$, and $O_3(G)$ satisfies one of the following:*

(a) *$O_3(G)$ is elementary abelian, G has a normal subgroup L, say, such that $O_3(G)=K\times L$, where $K\simeq L\simeq Z_3^3$;*

(b) *$O_3(G)=Z_9^3$ is homocyclic of exponent 9;*

(c) *$O_3(G)$ is the relatively free 3-generator, exponent 3, nilpotency class 2 group, of order 3^6, and $K=O_3(G)'=\Phi(O_3(G))=Z(O_3(G))$.*

Proof. The group G_α acts irreducibly on K and $O_3(G)/K$. If $O_3(G)$ is elementary abelian then, as $|G_\alpha|$ is relatively prime to 3, $O_3(G)$, regarded as a $\mathrm{GF}(3)G_\alpha$-module, is completely reducible by Maschke's Theorem. Hence, there is a normal subgroup L of G such that $O_3(G)=K\times L$ and (a) holds. Suppose next that $O_3(G)$ is abelian of exponent greater than 3. Then $O_3(G)$ has exponent 9 and it follows that $O_3(G)=Z_9^3$ (since G_α is irreducible on $O_3(G)/K$ and $K\subseteq\Omega_1(O_3(G))$). Thus, (b) is true.

Finally, suppose that $O_3(G)$ is nonabelian. Now the centre Z of $O_3(G)$ intersects K in a nontrivial G-invariant subgroup and, as G is irreducible on K, $K\subseteq Z$. Further, as G is irreducible on $O_3(G)/K$, it follows that $K=Z$. Similarly, $K=\Phi(O_3(G))=O_3(G)'$, and $O_3(G)$ has nilpotency class 2. Suppose now that some element g of $O_3(G)$ has order 9. Then $g\notin K$ and, as G_α is transitive on the 13 subgroups of $O_3(G)/K$ of order 3, it follows that each coset of K in $O_3(G)$ contains an element of order 9. As K is central in $O_3(G)$, it follows that every element of $O_3(G)-K$ has order 9. It follows that the automorphism group of $O_3(G)$ is transitive on the subgroups of $O_3(G)$ of order 3 and, hence, by [11, 12], $O_3(G)$ is abelian, which is a contradiction. Thus, $O_3(G)$ has exponent 3. Further, since the Frattini factor group of $O_3(G)$ has order 27, $O_3(G)$ is a 3-generator group. Finally, since the relatively free 3-generator, exponent 3, class 2 group has order 3^6, it follows that this group is isomorphic to $O_3(G)$. This completes the proof of Lemma 4.5. $\square$

Lemma 4.6. *If $O_3(G)$ is elementary abelian then $G_\alpha \simeq Z_{13}$.*

Proof. Suppose that $O_3(G) = K \times L$, as in Lemma 4.5(a), and suppose that $|G_\alpha| = 39$. Then G_α contains all of the 13 Sylow 3-subgroups of $\Gamma\mathrm{L}(1,27)$ and, hence, G_α contains a subgroup $\langle g \rangle$ of order 3 which acts on K and $O_3(G)/K$ (each identified with the additive group of GF(27)) as the group of field automorphisms of GF(27). Thus, g centralizes a subgroup of K of order 3 and a subgroup of $O_3(G)/K$ of order 3. If g centralizes $\langle hK \rangle$, where $h \in L - \{1\}$, then $h^g \in hK \cap L$ (since L is normal in G) and, hence, $h^g = h$. It follows that g centralizes 3 elements of each of K, hK and h^2K and, hence, g centralizes 9 elements of $O_3(G)$. Let $C_K(g) = \langle f \rangle$, so that $C_{O_3(G)}(g) = \langle h, f \rangle$. Let B be the block of $\mathscr{D}$ containing f and f^2. Then $G_B = \langle g \rangle$ and, as G preserves the equivalence relation τ, with equivalence classes the L-cosets in $O_3(G)$, B must contain a unique pair of τ-related points, that is, a unique pair of points in the same coset of L. Also G_B must fix this pair setwise and, hence, must fix the L-coset they belong to. However, the only cosets of L fixed by G_B are L, Lf and Lf^2, and the only pairs of points in these cosets left fixed by G_B consist of points in $\langle h, f \rangle$. On the other hand, G_B fixes setwise the set of 6 cosets of K which intersect B in a unique point. These 6 cosets must comprise two orbits of G_B on $O_3(G)/K$ of length 3 and, hence, B can contain no points from hK or h^2K. Thus, it is not possible for B to contain two points in $\langle h, f \rangle$ lying in the same L-coset. This is a contradiction. $\square$

Theorem 1.2 follows from the results proved in this section. We conclude with a brief discussion of the problem of determining the existence and the number of 2-$(729, 8, 1)$ designs $\mathscr{D}$ that exist. For each of the three possible groups $O_3(G)$ determined in Lemma 4.5, the group $\bar{G} = O_3(G)H$, where $H \simeq Z_{13}$, acts regularly on blocks and $\bar{G}$ has 28 orbits on unordered pairs of points, each of length $3^6 \cdot 13$. By [3, Proposition 1.3], the images under $\bar{G}$ of an 8-element subset B of $O_3(G)$ form the block set of a 2-design if and only if B contains exactly one unordered pair from each of the 28 orbits of $\bar{G}$ on unordered pairs. For any design $\mathscr{D}$, there is a block B containing a given pair of points α, β of K. The points α, β are the unique pair of ρ-related points in B and, so, the remaining six points of B come from 6 distinct nontrivial cosets of K in $O_3(G)$. These basic observations underly the computer searches which were carried out and are described in [10]. Exactly 467 designs were found.

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